Math In Spite of Itself: Quaternions, Piaget, Emerson, and Self Knowledge

Quaternion rendering

# Chapter 1: Math in Spite of Itself -- Motivation

Evans: Why do you think that mathematics is so important in the study of the development of knowledge?
Piaget: Because, along with its formal logic, mathematics is the only entirely deductive discipline. Everything in it stems from the subject's activity. It is man-made. What is interesting about physics is the relationship between the subject's activity and reality. What is interesting about mathematics is that it is the totality of what is possible. And of course the totality of what is possible is the subject's own creation. That is, unless one is a Platonist.

*From a 1973 interview with Richard Evans (Jean Piaget: The Man and His Ideas)*

Panksepp:…While I’m very attracted to these kinds of interpretations that root higher cognitive processes to basic bodily processes, I still feel that a deeper explanation can be found in the locomotive circuits because before there were parent/offspring relationships, there were predator and prey relationships, and there was the absolutely fundamental mandate of moving the body from point A to point B so as to intercept an object of attraction. This would also be more compatible with the research of Dr. Wolpert who states that everything about the brain is dependent on locomotion. Therefore I believe the emotional affects are not predicated on survival mechanisms or neurological circuitry hardwired in the Central Nervous System, but rather how the body and brain interplay on a much deeper level, specifically how to move the body from point A to point B. Emotional experiences arise in the mind when the  brain generates neurological energies by tracking a subliminal beam of attention on the body’s physical center of gravity, that are likewise coupled to the movement of  external objects of attraction.

From a 2013 interview of Jaak Pansepp posted by Robert Behan, titled “Music, Natural Dog Training, Panksepp and the Natural Law.”

## Is Math, By Its Nature, Controllable and Shapeable?

In my senior year of college as a math major at the University of Chicago, my undergraduate math department chairman, William H.L. Herman Meyer, invited me to tutor adults from outside the community who needed assistance in learning to “control math instead of fearing it.” I was able to do this successfully with a number of students, primarily through using analogies that appealed to them, and also through demonstrating to them that math was man-made, and there were a variety of ways or paths available to them to find the solution to a problem. This always came as a great surprise to them – they would say, “No, no, you have to be exact and follow the rules and procedures or otherwise you are wrong -- there is only one way to do things.”

But this is not the nature of math, as Piaget points out. First, within a given system, there numerous ways to express and represent the same problem or relationship – using a different coordinate system, or transformation path, or process perspective, for example. And the system itself can be extended, enlarged, transformed into a more elaborate and powerful system with the creation of artificial elements with new properties, so that several domains can be embraced and blended into a single system – such as the hypercomplex number systems – quaternions, octonions, and others --that Piaget found so helpful to express the processes and relationships of development.

Math has always contained many paths which have to be explored in order to understand the full meanings, implications, *and potential* of a mathematical object or system. And, as we said, whenever these prove inadequate, math can be extended in order to let us solve the problems posed to us. This is how the real numbers were extended to become the complex numbers (ordinary “imaginary numbers”), representing the solution to the puzzle of the relationship “i” squared equals -1. A new element “i” and a system to support it were born and found great use. This work was done early in the 1700s and early 1800s.

Out of this complex numbers system, even more elaborate math systems called quaternion math, were developed in the mid-1800s, in which several different kinds of imaginary numbers representing different dimensions were “created” – imagined and given properties and relationships that produced rules and techniques among other things expressing and executing rotations of objects in 3-D space. Today these systems are vital to the professional work of videogame programmers, astronautics engineers, and trackers of animal motions in which the body moves through a variety of positions in which it is pointing. (This is called *bio-logging*.) And from these, still further systems called octonions and Clifford algebras were defined and studied and applied. Collectively they are referred to as *hypercomplex numbers – indicating the presence of more than one kind of imaginary number in the system.*

Quaternions are also the system that Piaget seized upon “from the very beginning,” as he told me, to try to model the cognitive rules of the developing young mind, and, in some situations of confronting new fields, the adult mind. Piaget also used octonions, which contain the quaternions, to illustrate how nature uses imbedded systems (the old being included in the new and enlarged system) to retain older schemes of processing knowledge in their intact form. Octonions contain quaternions, which contain complex numbers, which contain real numbers. These suggestively correspond to Piaget’s four stages of cognitive processing, in terms of illustrating an imbedded set of systems.

The purpose of our book, for you, is not to learn quaternion math and work with it – there are a number of books that do that -- but rather to appreciate the efforts of those who do, across two centuries -- to understand the cultural relevance and especially the human cognitive research challenge of these systems. It is a surprisingly exciting and varied story, with many rewarding personalities along the way – and not just mathematicians.

*In a sense, math is always struggling with itself, through mathematicians, to create a larger, more encompassing version of itself, by analogy like a piece of software that develops new features to handle image processing, to supplement its existing text processing abilities. In the words of philosophers oriented to Plato and Aristotle respectively, math can be seen as both being and becoming. Benjamin Peirce, introduced below, understood math deeply in both these ways, using math as a foundation for things he considered certain and eternal, and also as wonderful developmental opportunity to explore and build new systems with quaternions and related tools together known as linear algebra. Today these systems are used to represent parts of the basic processes of matter and life.*

## Focus of This Book

About a hundred years ago, at the end of WWI, a 22-year-old Jean Piaget wrote an autobiographical novel called *Recherche* (Research) from which the above quote is taken. This novel was about understanding the basis of knowledge in the human brain as studied by Psychology. The technical term for this is epistemology – where does knowledge come from, and how do we know. In an age of increasing societal efforts in developing and applying “artificial intelligence,” the “how do we know” question has not gone away.

I believe that an important element that we need to better harness in studying these questions is **quaternion math and its cousins, octonion math and other “hypercomplex” math.**

As I have said, Piaget used hypercomplex math systems as a cornerstone of his cognitive developmental theory, using their structure. Where did he learn about these systems and their potential? It was from two highly remarkable father-and-son 19th-Century American mathematicians in the Harvard orbit, Benjamin and Charles Sanders Peirce. We will discuss this in greater detail later.

But this book is not primarily about Piaget (the giant of 20th-Century developmental-stages child psychology though he is a critical focal point and benchmark), or even primarily about math generally. Rather, this book is about describing and analyzing the **remarkable nature, variety, and potential of the quaternion family of mathematics**, primarily through examining its range of applications, as a way to tease out its underlying nature, its archetype, and to chart its future, especially in human motorics, vision, imagination, intelligence, and music.

This book consists of three interspersed activities –

* intellectual history connecting Piaget backward all the way to Goethe before 1800 and forward all the way to Artificial General Intelligence expert Ben Goertzel of today.
* enumeration and analysis of some of the most dramatic uses that quaternions have been put to in the levels of nature from quantum particles; to DNA coded strings; to animal bio-logging of body heading attitude; to cognitive development in humans.
* Considerations of consciousness, short-term memory, mirror neurons and interior selves, and music, as complex objects that quaternion math systems might be able to model, just as they did for cognition and development n the case of Piaget’s research.

Now let me ask rhetorically about what generated my interest in quaternions and later in Piaget and them? The answer is sketched below, beginning with how I became a mathematician, at least at the undergraduate level, and went on to a career in computers and, to some extent, cognitive processes.

## Carousel of Careers in Math and Computers

I never intended to become a mathematician or a computer professional or a psychologist.

As a child I wanted to be an adventurer or, later, a historian of American development and genius. Other possibilities also attracted me at the end of high school and of college– the Peace Corps, the Space Program, the Air Force Academy, or research physics, looking into the structure of matter.

I pursued all of these but chose the last one, physics at the University of Chicago, as my primary focus. I even worked in the physics department for two years. The field was interesting but not satisfying enough, and I didn’t excel in it in the way I did during high school.

After two years I switched to math, wanting to study the nature of structure itself, and to take abstract (pure) math courses that physics majors didn’t have time or direct purpose to take – especially linear algebra and projective geometry. I also felt that math could lead into a wider range of careers than physics could.

During one of these pure math courses, we were introduced for one day to a math system called quaternions that had been the rage in American 19th-Century math, but had been supplanted by matrix math because it was more streamlined. I had an interesting reaction. I imagined that perhaps there were new uses for quaternions in psychology, in studying the operations of the mind, just waiting to be discovered.

*Note: Thirteen years later, in graduate school, I learned that Piaget had done something of this kind a long time earlier, but few of his followers knew or understood this structural math based on three imaginary numbers. Some did know that this quaternion math was part of a larger field of math called group theory, but its meaning was vague to most non-mathematicians.*

After college I investigated several logical career paths leading from math – math teaching, math research, actuarial work in insurance, and computer programming with a specialty in operations research. I worked in insurance for a year, bringing me to New York – I’ve never left – and then began my long career in the computer field. Within several years (in 1966), I was programming a computer model for the construction of the 12-block-long foundation perimeter wall for the World Trade Center, and working on numerous other interesting projects.

About eight or sixteen generations of computers, large, medium, and then small, followed along, depending on how you count them. As a market analyst and forecaster, I helped shape the advancement of this market of smaller products by recognizing its coming unheard-of size and seeing and interpreting the reasons for its vast diversity and inclusiveness.

In 1970, when my computer work needed a deeper, more social and spiritual quality connecting my life to people and social concerns, I entered a special Ph.D. program designed for established computer professionals that looked at computer tools and education-and-training research together, studying the ideas of traditional psychologists like Jean Piaget and Jerome Bruner, and AI researchers like Marvin Minsky and Herbert Simon. I met and briefly talked with all of them when they were giving a guest lecture at or near the CUNY Graduate Center.

By 1977, I had completed a major project at CUNY/Baruch College, designing a nationally oriented Computer Center for Visually Impaired People. It still thrives under the enthusiastic wing and wisdom of Karen Luxton Gourgey, herself blind from birth. Many thousands of blind people have received computer educational and vocational services from them and experienced demonstrations of a range of types of equipment showcased there.

## The Engines of Thought

From 1970-1972 and again from 1979-1980 I became interested in applying quaternions to the representation of relationships in knowledge and language. My efforts felt promising for a while. I even created a “music mix” (a sequence of music tracks) that symbolized the qualities of various kinds of math that I had learned could represent various processes in cognitive psychology, especially quaternions. The title I applied to this music presentation was “The Engines of Thought” -- relational power that would lift the early human thinking enterprise off the ground and keep it propelled. And that is how I still feel about it.

But I put it away for many years. I ran into problems and became stuck, unable to develop my model and conjectures further, because my faculty did not have the background nor inclination to find a way to understand what I was trying to do, and to help me pursue and build on it. This was before the fMRI days of neuroscience, broad involvement in AI and robotics, and the development of the Internet to search out diverse examples of research with a common core, such as quaternions.

My involvement in the Ph.D. program ended in 1980, after several attempts at finding an acceptable doctoral dissertation subject. I went on to consult for a few years in technology for the blind, in which I came to appreciate the power of individual differences as important factors in computer use. After that, I moved into another computer sub-field of the computer profession, doing technical writing and IT documentation design for large global enterprises, including Deutsche Bank and Swiss Reinsurance, on challenging infrastructure integrity projects. I’m still engaged in this kind of work, in addition to my research and writing in the history and psychology fields, and in long-range social/technical analysis and speculative forecasting.

## My Early Hypotheses about Quaternions

Back in 1970, at the beginning of graduate school, I came to believe at least four things about quaternions:

* Quaternions and human intelligence may have developed in the brain as a structural offshoot of the need of proto-humans, who had come down off the trees to live on the ground, to mentally rotate objects in space in such a way as to recover the downward-looking picture in which the relationship of various objects was clearer.
* Some form of internalized quaternion transformation and control process may have helped to power the mental rotation and integrate it with other mathematically based systems, such as the projective geometry math that seems to transform curved 2-D images on our retina into 3-D “surround” images of the space around us, done for us by our retinoid system. (Octonion “parents “of imbedded quaternions happen also to contain the Fano plane, the simplest possible projective geometry system, consisting of 7 lines and 7 points, and self-duality-establishing point/line membership rules.)
* The quaternion math relationships, first established by the mental rotation system, may go on to be utilized by many other systems, such as language, to produce the skeleton of “general intelligence” processing. Spatial relations intelligence is a strong component of general intelligence testing.
* Language processing uses of quaternions may occur. If so, some of the reasons might be the economical and efficient use of short-term memory (STM) by supporting chunking and substituting of elements, a property of group theory, the larger category in which hypercomplex numbers reside. A distinct plus for this with respect to quaternion rules that quaternions are sensitive to order reversal – the sign of the product of two non-identical quaternions is reversed when their order is reversed.

I also learned, in about 1970/1971, that Piaget had pointed to a cousin of the quaternion group, the Klein 4-group (half a quaternion, with no negative elements, just the elements 1, i, j, k , in which each pair of “multiplied” letters produced the third letter, and the product of a letter with itself was 1). This group was also known as the INRC group (I for identity; N for negation; R for reciprocality; and C for correlative). Its four elements stood for transformations of logical propositions. N times R gives C, etc.

Here suddenly was an undeniable bridge between my former world of pure math and my emerging world of the psychology of knowledge and its structure and process.

## Meeting Piaget

I met Piaget a year later, in 1972 in Philadelphia, at the first annual meeting of the Jean Piaget Society. It was after a presentation given by his colleague, Barbel Inhelder, on four microstages occuring during the transition period between one major stage and another, and how this transition period could be speeded up.

I had already spoken with her after an earlier presentation in 1969 on memory, at the AERA annual conference (American Educational Research Association) in Los Angeles.

Now she introduced me to Piaget, who was getting his coffee between sessions. She referred to me as *le pauvre enfant* that had beenwaiting patiently for so long to talk to him. He obliged. I managed to ask him in my halting French (for I had had just a year of French in college, and it was directed to reading, not speaking) the revealing question that had sat inside me for a year, waiting to come out: “Did you ever explore quaternions in your work?”

He saw that my grasp of French was not great, and he answered simply, but also with much enthusiasm: “D’abord! This meant, because of the way he emphatically said it, “From the very beginning!” (The root of the French word refers to coming on board a ship). Then he turned away to his coffee.

I asked my question because I had known that he got his college education around the time of WWI – at a time not so long after the time in late 19th Century that quaternions had been held in high regard, and that his teachers or reading might reflect that and get him interested in exploring them. It turned out to be far more central an interest for him than even I could have imagined.

Piaget really cared about quaternions.

## Reawakening

Over 40 years later, my interest in quaternions came to life again because of my discussing music cognition and mental rotation at a Meetup event given by Marina Korsakova-Kreyn in New York City in the spring of 2013. After several sessions of discussions on cognition and the mathematical nature of quaternions, we became colleagues and I led a research effort that summer by the two of us to develop our insight further by intensive search for researchers and materials on different forms of quaternion applications, and a selective historical research on paths connecting Piaget to much earlier and much later quaternion and related research (e.g. the theory of relatives, the science of kinds, and artificial general intelligence). We also wrote to a dozen different researchers to share our interests.

[Expand above paragraph – our common ideas and principles]

## So What Does Our Book Title Really Mean?

*Math in Spite of Itself* refers to several basic conflicts within the nature of math as practiced as a research and modeling profession:

* The conflict between an established math system and its need for augmented elements and an extended system of rules.
* The conflict between pure math and applied math as an overriding value.
* The conflict between math as an abstract assumptions-based system versus math as a representation of some aspect of the real world as explained by physics and other subject fields. For a long time, the rate of adoption of quaternions as an object of interest and use was slowed down by the hesitation of many, including even its inventor, Hamilton, to regard these 4-D elements as “legitimate,” in the sense of being part of the physical world.
* The conflict between abstract pure math and “constructivist” (Bourbaki school) pure math over whether, for example, a mathematical object could just be arbitrarily defined in terms of attributes alone, or whether it also had to be proved that the object existed in a virtual sense, by demonstrating a construction process for the object. (This brings the development principle and value in front and center. Piaget was a strong supporter of the Bourbaki school of math critics wanting more substance and less arbitrary abstraction.)
* The conflict between the opposing philosophical/scientific views that math is invented, as Piaget says, versus that math is innate in nature in the physical world or in the human organism and mind, or even in a higher realm, as Emerson says, using Plato and the concept of ideals.

# Chapter 2: Method

What Matters – What Are the Big Questions?

A series of fundamental scientific and mathematical questions underlie my interest in the research of this book:

1. How are quaternions (which live in the algebra domain) connected to projective geometry (which lives in the geometry domain? How are they connected in mathematics (John Baez shows how they are organically connected within octonion structure), and how are they connected in living systems, literally in the biological world and in our human brain processes; and figuratively in linguistic and other symbolic processes, such as linguistics, AI, etc.;
2. How can questions of quaternion-based cognitive processing help resolve philosophy-of-science questions such as whether math is invented, innate, a blend of these, or neither?
3. What role does octonions above play in the operations and structuring of our short-term memory (STM)?
* Ben Goertzel believes that octonion math provides the structure of 7 degrees of freedom and the temporally reversible processing support for STM
* Herbert Simon believed that an algebraic group-theory approach to STM was probably true, but he didn’t know how to prove it.
1. How are quaternion-describable processes at one level in nature passed upward to a higher level of nature? How widespread is the opportunity to do this?
2. Can quaternions and octonions help to connect developmental processes with other cognitive and motor processes?
* Piaget and Inhelder’s research with hypercomplex numbers is consistent with this view.
1. Can quaternions be used to lead to new insights in music and to model significant portions of music cognition (*i.e. how music is processed and experienced in the brain*)?
* Marina Korsakova-Kreyn believes they can be used to do this, and such a view is consistent with her research on mental rotation of melodies.
1. Can quaternion and octonion models in music cognition help us to generalize our understanding of 4-D hyperspace cognitive transformational models championed by Karl Pribram for the past 30 years?
2. Do quaternions have a bright future in cognitive applications and in helping science to “go deeper” and be more unified and integrated?

We will undertake an extensive investigation of the variety of ways in which quaternions and their mathematical cousins are used to model processes and dynamics at each major level of nature – ranging from the fundamental physics of matter and energy described by Dirac and the Standard Model of physics, to the genome linguistics of DNA, to the developmental psychology of Jean Piaget.

Before doing this, we will trace a bit of intellectual history about quaternions as a model-building tool of logic and thought in the 19th Century.

This will take us from the 18th Century writer and scientist, Goethe, who shared Piaget’s view that science was just as much a science of qualities or attributes or kinds as it was a science of measurement, to Harvard mathematician Benjamin Peirce, who dreamed about imaginary numbers as a child and who worked extensively on developing the quaternion mathematics of combining three flavors of imaginary numbers as an adult, to his son, philosopher Charles Peirce, who invented *pragmatism* as aphilosophy and semiotics as a science of codes in and beyond language and developed the mathematical *theory of relatives* (or relations), to Piaget himself.

We will also trace the connection of Goethe to the brilliant educator, Rudolf Steiner, who trained teachers in the principles of biological and human development, beginning with projective geometry.

We will then discuss how Ben Goertzel extracted a model of the switching control of interior selves based on the concept of mirror neurons and Piaget’s use of quaternion and octonion mathematics.

## Relationships Between Several Key Historical Figures and With Mathematics,

From Goethe to Goertzel.



## Quaternions and Octonions

What are quaternions? – They are a fairly simple type of system mathematical and objects:

* The quaternion group contains 8 elements: 1, i, j, k and their negatives, -1, -i, -j, -k . The numbers designated by letters (i, j, k) are all imaginary numbers, but each different from each other.
* Quaternions are a 4-D system of blends of real and complex numbers, with a unifying group structure that is affected by operation order, unlike most number systems.
* Quaternions are characterized as non-commutative (i.e. affected by switching around the order of multiplying them, so ij is *not* ji -- rather it is *minus* ji. Also, I square equals -1, so does j square and k square.

The rich relations present between these elements enables them to model rotations in spaces of all kinds, and to determine attitude (i.e. pointing direction during motion). Software applications have been written to do this for animal motion research (bio-logging), space shuttle maneuvers, and “molecular docking” in the context of pharmaceutical research. Quaternions capture the essence of turning and pointing anywhere in nature.

Quaternions are also a component of a larger 8-D algebraic system called octonions, also discovered in 1843. They also have a simple sister component (overlapping with 7 quaternions) inside octonions called the Fano projective plane consisting of 7 lines and 7 points, with each line containing 3 points.

Quaternions and octonions can represent and be used to conveniently calculate 4-D rotation operations in space (e.g. as in video game screen displays), as well as represent relationships in:

* communication transmission
* information storage
* color structure
* logical proposition transformations
* child development stages with nested schemas at each stage
* multiple mirror neuron reflections represent multiple interior selves
* quantum physics entities and processes
* Bio-logging (capturing animal movements through sensors by determining attitude of the body in relation to reference lines wirelessly obtained
* and (via embedding in octonion algebra) they assist in the conceptual theoretical modeling of the structure of short-term human memory.

*The list above is longer and more varied than one might think.*

Quaternions were discovered/invented in 1843, just 11 years after the German poet and scientist Goethe died. In the intervening years since then, quaternions have experienced years of fame (roughly until 1905) and years of marginality (until the current decade).

Where does the quaternion fit into the universe? And what are the quaternion implications for understanding human cognition and the organization of nature? The answer is that their set of roles seem vast and comprehensive. (See Table 1).

There are a number of meanings that can be attached to this question. Taking my inspiration from Goethe’s delicate empiricism approach to research methodology, we will look at as many of them as we can, to get many perspectives from the effort, and thus be in a position to see, to recognize “the archetypal quaternion presence and use.”

Table 1: Levels of Nature, and Quaternion Application Examples

| Level | Theme | Reference |
| --- | --- | --- |
| Fundamental Physics | Particle physics in quantum theory --Dirac formulations of fundamental laws using entities based on quaternion concepts | Dixon (1994) – see Ben Goertzel’s Mirrorhouse paper  |
| Mathematical Physics | Using quaternions to model most phenomena occurring in physics. Exception – angular momentum in a gravity field – results are dependent on the coordinate system. | Doug Sweetser website:[www.quaternions.com](http://www.quaternions.com)See also his videos showing pipe-cleaner models of hypercomplex structures.Doug states that “quaternions are the ‘double-entry’ bookkeeping system for the universe – more than one way to do” each and every thing. |
| Cell Biology | Genome/DNA Linguistics | Sergey Petoukhov (2011) |
| Human Language | **Quantum Bose-Einstein (sociable) statistics** for word frequency (relatable to quaternions, I think)**Projective Geometry**--Chinese Grammar (part-of-speech role transformations of all words) | Gustav Herdan (1962) – Calculus of Linguistic Observations |
| Human Cognition | Cognitive Development (schemata nesting at each step – octonions, quaternions, complex numbers, reals) and Logical Proposition Transformations ([Felix] Klein-4 group --- 1,i,j,k)  i-square=-1 etc | Psychology and the Child (2001), by Piaget & InhelderAlso see excerpt from attached article on Epistemology & Psychology of Functions, concerning quaternion framework for categories construction by the child and adults.See slide below on Genetic Epistemology – 4 nested systems in octonions probably correspond the Piaget’s 4 cognitive development stages with nested schemata – they at least meet a necessary condition. |
| Cell Biology | Genome/DNA Linguistics | Sergey Petoukhov (2011) |
| Human Social Processing and Roles | Mirrorhouse paperMirror Neurons – algebraic  ensemble design for a collection of internal Self Entities. | Mirror Neurons, Mirrorhouses, and the Algebraic Structure of the Self (2008)<http://www.goertzel.org/dynapsyc/2007/mirrorself.pdf> |
| Prehistory ResearchCognitive, developmental, and aesthetic | Religious expression in animal-depicting artwork of 10,000 12,000BC | [http://www.originsnet.org/upgallery1animals/pages/n)niauxquaternion.htm](http://www.originsnet.org/upgallery1animals/pages/n%29niauxquaternion.htm) NIAUX IMAGE[http://www.originsnet.org/upgallery1animals/pages/o)portalquaternion.htm](http://www.originsnet.org/upgallery1animals/pages/o%29portalquaternion.htm) PORTAL IMAGEBy James HerrodJames’ research here is based on complementarity among four images, a quaternion-inspired process keyed to Piaget and Levi-Strauss.Prehistory hominins of much older times (over a million years ago) made heavy use ofphysical and mental rotation in tool use and in perceiving animals.We should investigate prehistory contexts relative to various possible cognitive uses of quaternions, and their evolution. |

## Why Refer To Goethe? – The Role of the Archetype (Seeing the Object from many Viewpoints)

What is the “Soul” of the quaternion – what is its essential nature? Not defining nature in the sense of formula, but perceived nature in the light of history of its properties, relations, and especially applications. In this light:

* Jacob Needleman spoke of The Soul of the American People.
* Tracy Kidder spoke of The Soul of a New Machine.

These objects are both infused with the experience of a group and include a perception of a realized essence from that experience.

In that spirit, we look to find the essential nature of quaternions and their cousins in hypercomplex mathematics, the octonions.

Goethe explained that to investigate a subject well, you had to investigate it from all possible angles and points of view in order to catch the range of its behavior under different conditions. Then you could begin to generalize, and to characterize a representable example or form, which he called an archetype.

The archetype is roughly comparable to pulling out the thematic essence in concrete form in a poetic context, after much exposure, either in the poetic creating phase or in a reader’s receiving and digesting phase. This is a process Goethe knew much about. It also resonates with Poincare’s picture of scientific creativity, which we will say more about.

## Varieties of Philosophical Contexts

In describing the various kinds of applications and interpretations of quaternions, rather than finding one uniform overarching philosophy, we will find a number of divergent beds of philosophy in which this math use has been placed. A sample of the philosophies and project /thinking frameworks are given below:

* Non-traditional philosophy – Goethe
* Neo-Platonic – Emerson and Benjamin Peirce
* Pragmatism[Natural logic] – Charles Peirce
* Structuralism – Piaget
* Holistic – Karl Pribram
* Artificial General Intelligence (AGI) – Ben Goertzel

## Key Themes and Questions

### What Characteristics Might An Archetypal Quaternion Application Display?

* The **quality of turning**, stemming from rotation process control.
	+ Stemming from this, the implementation of **turning points.**
* **The quality of transforming** of objects**.**
	+ Stemming from this, the implementation of **creating/changing states**.
	+ Ben Goertzel’s Mirrorself/Mirrorhouse paper reflects this, in switching in and out of different selves in states.
* The **quality of controlling the direction of flow** – as in the switching and routing of packets on the Internet

# Chapter 3: Across the Centuries

Our story involves historical figures spanning the last 230 years, in both America and Europe. Mostly they are researchers, but some are also from cultural and legal and even music fields.

These include two categories of people: (1) researchers and philosophers, and (2) mathematicians.

We will enumerate a healthy cross-section of them below. But first let us look at several key connections across time:

* Goethe, born in 1749, had similar views to Piaget, born in 1896, concerning the need to balance the emphasis on measurement in science with a *strong attention to qualities or attributes or kinds* – to look in detail into the various dimensions that objects differed from each other or were similar, so that familiarity with these conditions could be treated appropriately in experiments and in thinking about the behavior of the objects. His approach connected back to Aristotle and attributes, for example, of plant classification. In terms of motion of the subject, he said symbolically that the researcher must be as “agile and mobile” as his subject.
* Emerson was inspired by Goethe’s carefully and thoroughly observational approach to science and nature. He in turn convinced Benjamin Peirce, who was to become the greatest American mathematician of the 19th Century, to put his efforts in the direction of pure math rather than the prevailing focus on applied math and practical results. He used a Platonic approach to argue that the resulting foundation aiming toward derivation and proof – a form of certainty given a set of assumptions -- would lay deeper, go farther, and be more permanent than simply a practical emphasis.
* Benjamin Peirce developed quaternion theory in its early stages after it was invented by William Hamilton. Peirce was an unexpected strong supporter of Hamilton across the ocean, along with his student, Thomas Hill, who later became president of Harvard. Peirce was math chairman at Harvard. Peirce developed the comprehensive theory of linear algebra, a pure math subject at the time. Hamilton and Peirce knew that quaternion elements could be represented in coordinate system form as points/locations expressed as ordered quadruples in a 4-D space -- e.g. (1,0,0,1) or (1,1,1,1).
* Charles Sanders Peirce, Benjamin’s son, developed these coordinate points in “4-space” into a quaternion treatment of a “theory of relatives” (relations) to express selective role relationships between pairs of people like student and teacher, student and classmate, and teacher and colleague.
* Piaget: In his book, *Epistemology and Psychology of Functions, clarifying the meaning of functions,* Piaget quoted this above work by C.S. Peirce and his father’s work on regarding ordered quadruples as a form of quaternion. This was in the context of discussing the development of categories.

*Time Scope*: So in terms of (1) the scientific focus on attributes and (2) the use of quaternions as a conceptual tool, we see there is a definite line of evolution and development, or resonance in some cases, from the time of Goethe to the time of Piaget about 140 years later. In turn, the knowledge and algebra research work of AI/mathematics researcher BenGoertzel and of neuroscience researchers Karl Pribram and Antonio Damasio came in the period about 50-90 years after Piaget first worked with it.

Let me first present several corresponding reference tables of relevant historical figures and professional leaders for reference, before choosing a few (names bolded below), starting with Emerson, who in their personal stories are on the main line of the intellectual history of direct and indirect influences we want to trace -- for them we will give a larger descriptive development. These are Goethe, Emerson, the Peirces (father and son), Piaget, Pribram, Damasio, and Goertzel.

## Historical Figures and Professional Leaders

1. **Researchers/writers/philosophers**

| **Name** | **Field(s)** | **Dates** | **Comments** |
| --- | --- | --- | --- |
| **Wolfgang von Goethe** | Scientist and literary writer(German) | 1749-1832 | Was the greatest figure in German literature.In science, he studied and managed the mines and forest of the lands of the ruler of Weimar, and developed superior methods of research. |
| **Ralph Waldo Emerson** | Lecturer, poet, philosopher, minister, literary and social critic(U.S.) | 1803-1882 | The first man in American culture to make his living purely as a lecturer and then writer, based on the lectures. He had no regular pulpit or faculty or civic position. He was America’s first world cultural figure. In turn he helped Benjamin Peirce to become the equivalent person in world mathematics.Over the course of his lifetime, He reformed the theology and outlook of the Puritan-derived Unitarian religion in America. |
| **Rudolf Steiner** | Educator, educational theorist, epistemology, philosophy of science, spiritual science, Waldorf education(Austria) | 1861-1925 | Projective Geometry interpretations of biological development processes*Wikipedia*: Steiner advocated a form of ethical [individualism](http://en.wikipedia.org/wiki/Individualism), to which he later brought a more explicitly spiritual component. He based his [epistemology](http://en.wikipedia.org/wiki/Epistemology) on [Johann Wolfgang Goethe](http://en.wikipedia.org/wiki/Johann_Wolfgang_Goethe)'s world view, in which “Thinking … is no more and no less an organ of perception than the eye or ear. Just as the eye perceives colours and the ear sounds, so thinking perceives ideas.".[[9]](http://en.wikipedia.org/wiki/Rudolf_steiner#cite_note-9) A consistent thread that runs from his earliest philosophical phase through his later spiritual orientation is the goal of demonstrating that there are no essential limits to human knowledge.[[10]](http://en.wikipedia.org/wiki/Rudolf_steiner#cite_note-10) |
| Henri Poincare | Mathematical physicist, philosophy of science(France) | 1854-1912 | Examined the role of intuition in science development, using an introspective technique. |
| Paul Dirac | Physicist(British) | 1902-1984 | Quaternion applications to quantum physics model. |
| **Jean Piaget** | Developmental psychologist, epistemologist (nature of knowledge)(Swiss) | 1896-1980 | Nature of knowledge.Development of cognitive processes and structures. |
| Barbel Inhelder | Developmental psychologist, epistemologist (nature of knowledge)(Swiss) | 1913-1997 | Clinical treatment of special-needs children – techniques for acceleration of general cognitive developmentShe was Piaget’s research partner.*www.Intelltheory.com*: In contrast to her mentor and collaborator, Piaget, Inhelder took a much more process-oriented, functional approach to understanding thinking. The major change in thinking that occurs between childhood and adolescence involves the emergence of experimental or inductive thinking… During a visiting appointment at Harvard University in 1961-1962, Inhelder was able to break away from the logical-structural approach of Piaget and focus on applying the functional approach to genetic epistemology. Several studies were conducted in an attempt to better understand the transition between various stages of development. The work of her and her collaborators culminated in a book entitled Learning and the Development of Cognition. They found that when children were in a transition between stages, training the children on concepts related to the next stage of development was successful in facilitating the promotion to the next stage.     Although Inhelder was pleased with the results regarding training, she was more interested in the fact that those studies also helped clarify the mechanisms for transition between stages. This finding became the foundation for Inhelder's work on cognitive strategies and particularly children's development of problem solving abilities. Instead of looking at broad, general strategies used by children, she again took a process-oriented approach and examined how and when children used specific procedures (which could be thought to add up to a strategy). This work provides information on the link between functional knowledge (know how) and structural knowledge (know that), which has been further explored by many cognitive psychologists. |
| Olive Whicher | Biological processes and anthrosophic spiritual science (Rudolf Steiner)(British) | 1910-2006 | Projective Geometry interpretations of biological development processes. Book: Heart of the Matter: Discovering the Laws of Living OrganismsBook: Projective Geometry: Creative Polarities in Space and TimeLink:[Projective geometry; creative polarities in space and time - Olive ...](http://books.google.com/books/about/Projective_geometry_creative_polarities.html?id=bzoZAQAAIAAJ) books.google.com › [Mathematics](http://www.google.com/search?tbo=p&tbm=bks&q=subject:%22Mathematics%22) › [Geometry](http://www.google.com/search?tbo=p&tbm=bks&q=subject:%22Mathematics+Geometry%22) › [General](http://www.google.com/search?tbo=p&tbm=bks&q=subject:%22Mathematics+Geometry+General%22)‎Projective geometry; creative polarities in space and time. Front Cover. **Olive Whicher**. Rudolf Steiner Press, Jan 1, 1971 - Mathematics - 292 pages. |
| Herbert Simon | Cognitive scientist, organizational theory, AI | 1916-2001 | Nobel Prize in Economics, 1977, Limits of Rational Behavior |
| Rupert Sheldrake | Plant biologist, writer on fundamental issues of method and culture in science | 1942- |  |
| Seymour Papert | Mathematician, AI researcher, educational reformer | 1928- | Invented neural net concept.Co-invented Logo software Language for children.Collaborated with Piaget at Piaget’s Institute in Geneva (1958-1963).*Wikipedia:* Papert worked with Jean Piaget at the University of Geneva from 1958 to 1963[[4]](http://en.wikipedia.org/wiki/Seymour_Papert#cite_note-4) and is widely considered the most brilliant and successful of Piaget's protégés; Piaget once said that "no one understands my ideas as well as Papert." Papert has rethought how schools should work based on these theories of learning. |
| **Karl Pribram** | Neuropsychologist | 1919- | Created the holonomic/holographic brain theory. |
| E.H. Carlton | Psychologist | c. 1950s- | Colleague of Pribram’s who used quaternion math to create models of the visual system processing. |
| Antonio Damasio | Neuroscientist and clinician | 1944- | Neuroscience researcher.Book: Self Comes To Mind (2012)Wikipedia: Damasio formulated the [somatic marker hypothesis](http://en.wikipedia.org/wiki/Somatic_marker_hypothesis),[[3]](http://en.wikipedia.org/wiki/Antonio_Damasio#cite_note-3) a theory about how emotions and their biological underpinnings are involved in decision-making (both positively and negatively, and often non-consciously). Emotions provide the scaffolding for the construction of[social cognition](http://en.wikipedia.org/wiki/Social_cognition) and are required for the self processes which undergird consciousness.[[*citation needed*](http://en.wikipedia.org/wiki/Wikipedia%3ACitation_needed)] "Damasio provides a contemporary scientific validation of the linkage between feelings and the body by highlighting the connection between mind and nerve cells...this personalized embodiment of mind."[[4]](http://en.wikipedia.org/wiki/Antonio_Damasio#cite_note-4) |
| David Lewin | Music theorist, composer, and critic -- he used mathematical models from projective geometry and group thory | 1933-2003 | *Wikipedia:***David Lewin** (July 2, 1933–May 5, 2003) was an American [music theorist](http://en.wikipedia.org/wiki/Music_theorist), music [critic](http://en.wikipedia.org/wiki/Critic) and [composer](http://en.wikipedia.org/wiki/Composer). Called "the most original and far-ranging theorist of his generation" (Cohn 2001), he did his most influential theoretical work on the development of [transformational theory](http://en.wikipedia.org/wiki/Transformational_theory), which involves the application of mathematical [group theory](http://en.wikipedia.org/wiki/Group_theory) to music…Important writings for the discipline of music theory include "Behind the Beyond" (1968–9), a response to [Edward Cone](http://en.wikipedia.org/wiki/Edward_Cone), and "Music Theory, Phenomenology, and Modes of Perception" (1986).*Music Article*: Some compositional uses of projective geometry.*Article by E. Gollin (2004)- Link:*[On David Lewin's "Projective Geometry" and the Aesthetic of ... - jstor](https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=1&ved=0CCwQFjAA&url=http%3A%2F%2Fwww.jstor.org%2Fstable%2F25164551&ei=sku-UvOTJI_JsQT4vILIDA&usg=AFQjCNFjZPMLP3AxIYrxdRv1asRJwOFqjA&sig2=vSYLbV25LdMeEZ3plLy-pQ)www.jstor.org/stable/25164551‎by E Gollin - ‎2004 |
| **Ben Goertzel** |  | 1966- | Chairman of the Artificial General Intelligence Society.Chief Scientist of financial prediction firm Aidyia Holdings.Chairman of AI software company Novamente LLC. |

1. **Mathematicians and related professionals**

| **Name** | **Field** | **Dates** | **Comments** |
| --- | --- | --- | --- |
| Carl Friedrich Gauss | Mathematician – Many topics, including complex numbers(Germany) | 1777-1855 | Developed the representational concept of the Complex Plane (using complex numbers):*Wikipedia:* It was not until 1831 that he overcame these doubts and published his treatise on complex numbers as points in the plane, largely establishing modern notation and terminology. The English mathematician [G. H. Hardy](http://en.wikipedia.org/wiki/G._H._Hardy) remarked that Gauss was the first mathematician to use complex numbers in 'a really confident and scientific way'  |
| William Hamilton | Irish mathematician, astronomer and physicist | 1905-1865 | Invented quaternion math (1843).*See note 1 at end of table.* |
| John T. Graves | Irish mathematician and jurist | 1806-1870 | Invented octonion math (1843), friend of Hamilton’s. |
| **Benjamin Peirce** | Mathematician – Algebra(U.S.) | 1809-1880 | Father of Charles Sanders Peirce.Chairman of Harvard Math Dept., developed the field of quaternions beginning five years after its invention and developed linear algebra, the larger field containing quaternions.*Wikipedia:* Benjamin Peirce is often regarded as the earliest American scientist whose research was recognized as world class.[[3]](http://en.wikipedia.org/wiki/Benjamin_Peirce#cite_note-3) |
| Arthur Cayley | Mathematician – Algebra, lawyer(British) | 1821-1895 | Using matrix theory, created a new foundation for classifying quaternion-like objects (Cayley-Dickson construction).Defined the modern binary-operation concept of group theory.Advocate and professor for pure math.Wikipedia: He helped found the modern British school of [pure mathematics](http://en.wikipedia.org/wiki/Pure_mathematics). |
| Hermann Grassmann | Mathematician – Geometric AlgebraPhysicist, neohumanist, linguist, publisher(Germany) | 1809-1877 | Invented geometric algebra.*Wikipedia*: For an introduction to the role of Grassmann's work in contemporary [mathematical physics](http://en.wikipedia.org/wiki/Mathematical_physics) see [*The Road to Reality*](http://en.wikipedia.org/wiki/The_Road_to_Reality)[[9]](http://en.wikipedia.org/wiki/Hermann_Grassmann#cite_note-9) by [Roger Penrose](http://en.wikipedia.org/wiki/Roger_Penrose). |
| William Clifford | Mathematician – Algebra, philosopher(British) | 1845-1879 |  |
| **Charles Sanders Peirce** | Logician, philosopher, linguist (invented semiotics)theorist), mathematician (U.S.) | 1839-1914 | Invented Pragmatics (philosophy).Invented semiotics (codes and language). |
| Felix Klein | (Germany) | 1849-1925 | Developed the Erlangen Programme for organizing algebra and geometry by means of invariants of transformations. |
| David Hilbert | Mathematician – Geometry(Germany) | 1862-1943 | Developed abstract geometry.Developed Hilbert Spaces (infinite-dimensional spaces). |
| Gustav Herdan | Quantitative linguist, statistical mechanics, epidemiology(British) | 1897-1968 | Master of Mathematical linguistics – applied quantum theory statistics, projective geometry, etc. |
| Buckminster Fuller | Designer, mathematician, social and economic critic(U.S.) | 1895-1983 |  |
| David Hestenes | Geometric Algebra, Mathematical Physics, science writing and education(U.S.) | 1933- | Hestenes emphasizes the important role of the mathematician [Hermann Grassmann](http://en.wikipedia.org/wiki/Hermann_Grassmann)[[25]](http://en.wikipedia.org/wiki/David_Hestenes#cite_note-25)[[26]](http://en.wikipedia.org/wiki/David_Hestenes#cite_note-26) for the development of geometric algebra, with [William Kingdon Clifford](http://en.wikipedia.org/wiki/William_Kingdon_Clifford) building on Grassmann's work. Hestenes is adamant about calling this mathematical approach “geometric algebra” and its extension “geometric calculus,” rather than referring to it as “Clifford algebra”. He emphasizes the universality of this approach, the foundations of which were laid by both Grassmann and Clifford. He points out that contributions were made by many individuals, and Clifford himself used the term “geometric algebra” which reflects the fact that this approach can be understood as a mathematical formulation of geometry, whereas, so Hestenes asserts, the term “Clifford algebra” is often regarded as simply “just one more algebra among many other algebras”,[[27]](http://en.wikipedia.org/wiki/David_Hestenes#cite_note-27) which withdraws attention from its role as a unified [language](http://en.wikipedia.org/wiki/Language) for mathematics and physics. |
| Doug Sweetser | Mathematical Physics(U.S.) | c. 1960- |  |
| John Baez | Mathematician-Algebra and Mathematical Physics(U.S.) | 1961- | Popularizes the nature and uses of octonions.Higher Category Theory. |

Notes:

1. [Peter Guthrie Tait](http://en.wikipedia.org/wiki/Peter_Guthrie_Tait) among others, advocated the use of Hamilton's quaternions. They were made a mandatory examination topic in Dublin, and for a while they were the only advanced mathematics taught in some American universities. However, controversy about the use of quaternions grew in the late 19th century. Some of Hamilton's supporters vociferously opposed the growing fields of vector algebra and vector calculus (from developers like [Oliver Heaviside](http://en.wikipedia.org/wiki/Oliver_Heaviside) and [Josiah Willard Gibbs](http://en.wikipedia.org/wiki/Josiah_Willard_Gibbs)), because quaternions provide superior notation. While this is undeniable for four dimensions, quaternions cannot be used with arbitrary dimensionality (though extensions like [Clifford algebras](http://en.wikipedia.org/wiki/Clifford_algebra) can). Vector notation had largely replaced the "[*space-time*](http://en.wikipedia.org/wiki/Space-time)" quaternions in science and engineering by the mid-20th century. *(from Wikipedia, William Rowan Hamilton)*
2. Today, the quaternions are used in [computer graphics](http://en.wikipedia.org/wiki/Computer_graphics), [control theory](http://en.wikipedia.org/wiki/Control_theory), [signal processing](http://en.wikipedia.org/wiki/Signal_processing), and orbital mechanics, mainly for representing rotations/orientations. For example, it is common for spacecraft attitude-control systems to be commanded in terms of quaternions, which are also used to telemeter their current attitude. The rationale is that combining many quaternion transformations is more numerically stable than combining many matrix transformations. In pure mathematics, quaternions show up significantly as one of the four finite-dimensional [normed division algebras](http://en.wikipedia.org/wiki/Normed_division_algebra) over the real numbers, with applications throughout algebra and geometry. *(from Wikipedia, William Rowan Hamilton)*

# Chapter 4: Emerson

## The Relevance of Emerson

Ralph Waldo Emerson, American 19th Century literary and spiritual giant, is both a player and a point of reference in our story of quaternion and pure math development and use in many fields, particularly in Piaget’s theory of intellectual development and the nature of knowledge.

“The nature of knowledge” ties in with our accompanying theme – self-knowledge to that of the nature of quaternions in this book. We will go into more detail on this later.

## Self Knowledge

Let us first address the role of self-knowledge.

Emerson urged young writer-graduates from college to pursue their own paths of creation and truth rather than an established standard one that urged you to only follow models of style and structure that were wrought by past masters such as Homer, Shakespeare, Goethe, and so forth (this was the European model of training and work in culture and the arts before Emerson’s time) instead of choosing and using forms that better fit your artistic need.

In mathematics, we might say to pick a coordinate system of representation for your principal objects that “went with the grain” and that was natural to it. This idea is also in harmony with Goethe’s principle of *delicate empiricism* -- of getting to know the real nature of your scientific subject, view it under all conditions and see it from all angles, so as to be able to create useful experiments and study it more deeply and accurately.

This implies a certain level of self-knowledge on the part of the young writer, which will grow with time as he/she develops his subject based partly on following his/her own intuitions to see where they lead. This becomes a *quest,* gathering resources from inside and out.

Emerson started professional life as a Unitarian minister in Boston around 1830, serving New England congregations such as the one in Concord that his grandfather served, which had formerly been Puritan until they voted to become Unitarian, a more liberal denomination, in the period 1810-1820. Later, he became a public lecturer, a poet, and a philosopher, writing on many subjects. My favorite is his early essay, *Circles,* about how we grow our horizons and self-identity*.*

Emerson had attended the non-denominational very liberal Harvard Divinity School in the 1820s. His teachers and colleagues were by and large religious liberals who were leading a portion of the Church of the Standing Order in New England (the Calvinist-oriented Puritan movement) into new religious territory.

When Emerson approached Benjamin Peirce to emphasize the importance of pure math over applied math, he used attitudes and principles from within this revolutionary set of religious attitudes. He showed Peirce how pure math could sit on a foundation of what Peirce took to be intellectual/spiritual certainty derived from Platonic ideas about absolutes (ideal objects), and other sources in Buddhist and Eastern thinking, ideas about how God was in Man and Man was in God.

I’m a Unitarian myself, and, several years ago, coordinated a discussion series on Emerson – an Emerson Circle -- and his relationship to other figures, including John Dewey, Friedrich Nietzche, Walt Whitman, and Adam Mickiewicz (Poland). Each month, each of these topics was researched by a different contributing member of the Circle. Out of this I began to see the many impacts that Emerson made to the young, emerging American culture, and the attention he drew, justifying young adults in seeking their own way.

So Emerson was doing for the development of American literary culture what his teachers at Harvard Divinity School were doing for the evolution of New England religious culture.

He was also helping to supply the intellectual/philosophical foundations (a *do it yourself* philosophy in contrast with Immanuel Kant’s elaborate intellectual structure) for the many national liberation movements in Europe – Polish, Irish, Spanish, etc. This was the reason for the interest the great Polish poet and leader Mickiewicz showed in him, while in exile in Paris in 1843, when he gave a series of public lectures relating to Emerson’s writings, making them known on the European Continent for the first time. He also developed a friendship with Emerson’s best friend and intellectual equal, Margaret Fuller, who visited him.

Relatively few people know of Emerson’s influence on Eastern Europe. Even fewer know of Emerson’s impact on American mathematics, through his influence on one of America’s earliest and greatest mathematicians, Benjamin Peirce (pronounced PURSE).

## Emerson and Benjamin Peirce

Daniel J. Cohen wrote a book in 2007 about the perception of Pure Math by the Victorian mind. It is titled Equations from God: Pure Mathematics and Victorian Faith. Chapter Two, “God and Math at Harvard: Benjamin Peirce and the Divinity of Mathematics,” looks at Emerson’s philosophical and spiritual influence over Benjamin Peirce over the course of several decades:

* First by Emerson’s famous address to the Harvard Divinity School Graduation in 1838.
* Then for the next several decades through the regular Cambridge-based Saturday Club discussions they attended together.

### From the Section Entitled Unitarianism and Transcendentalism in Daniel J. Cohen’s Book, Equations from God

[copy 3 pages of text excerpts]

## Emerson and Goethe

Let us take a step back and look at some of Goethe’s influences on Emerson.

**[Research some quotes, and comment on them.]**

**Use 3 pages of excerpts (from region of p 96-100, 245-249) from Robert D. Richardson, Jr. (1995). Emerson: The Mind On Fire, Berkeley: U. of Calif. Press**

### Goethe

Emerson turned to reading Goethe for solace when that great need was there. It happened often in his life – the loss of two dear grown brothers, the loss of a young wife, the loss of a boy.

Goethe seemed to deal with every human condition and personal character in his literary career, mirroring his approach to science. He was also a great scientist, but this clear fact has been obscured by the process of history. In America, we forget his scientific accomplishments. In Germany we remember he was a scientist but forget his scientific integrity and strength because of a heated, even nasty exchange of letters regarding Newton and his light experiments (Newton used only one set of experimental conditions and extrapolated many claims from it).

# Chapter 4A: En Route Meaning Check -- Interpreting “Life Paths,” through Emerson, Goethe, Piaget, Panksepp, and Quaternions

As an en-route check (toward the role of Self-Knowledge) for the meaning of what we have discussed and developed so far, we will look at several concepts:

* Jaak Panksepp interpretation of uses of emotion in locomotion
* The observer and the world – Piaget, Emerson, Goethe
* Humanity’s task for the individual – what this means (Emerson, Goethe, Hamilton); Dewey and Democracy; Deep History
* Quaternion direction-finding, NPR broadcast on direction-remembering cognition based on turning
* One’s direction in finding and doing what one is able to do at a high level and seeks to do – establishing, maintaining, recovering
* Religious interpretations of paths (Judeo-Christian, Buddhist, Native American), the Jonah story

QUOTE Emerson from Library of Philosophy – the individual; add in section from Emerson the rationalist – the Harvard “University Lectures”

QUOTE Emerson – Goethe’s view – role of art – the “Science” of making a Life (Aristotle & Virtue – what we are each perfected to do) -- meant to do by reason of our attributes – Self-Knowledge discovery

# Chapter 5: Goethe

*This is from the Rudolf Steiner Archives, on the inadequacy of understanding nature simply from writings of philosophers, such as Plato, rather than becoming involved with nature:*

Rudolf Steiner on Goethe:

…When he observed nature, it then brought ideas to meet him. He therefore could only think it to be filled with ideas. A world of ideas, which does not permeate the things of nature, which does not bring forth their appearing and disappearing, their becoming and growing, is for him a powerless web of thoughts. The logical spinning out of lines of thought, without descending into the real life and creative activity of nature seems to him unfruitful. For he feels himself intimately intertwined with nature. He regards himself as a living pan of nature. What arises within his spirit, according to his view, nature has allowed to arise within him. Man should not place himself in some corner and believe that he could there spin out of himself a web of thoughts which explains the being of things. He should continuously let the stream of world happening flow through himself. Then he will feel that the world of ideas is nothing other than the creative and active power of nature. He will not want to stand above the things in order to think about them, but rather he will delve into their depths and raise out of them what lives and works within them.

Goethe's artistic nature led him to this way of thinking. He felt his poetic creations grow forth out of his personality with the same necessity with which a flower blossoms. The way the spirit brought forth a work of art in him seemed to him to be no different than the way nature produces its creations. And as in the work of art the spiritual element is inseparable from its spiritless material, so also it was impossible for him, with a thing of nature, to picture the perception without the idea. A view therefore seemed foreign to him which saw in a perception only something unclear, confused, and which wanted to regard the world of ideas as separate and cleansed of all experience…

Therefore he could not find in the philosophers what he sought from them. He sought the ideas which live in the things and which let all the single things of experience appear as though growing forth out of a living whole, and the philosophers provided him with thought hulls which they had tied together into systems according to logical principles. Again and again he found himself thrown back upon himself when he sought from others the explanations to the riddles with which nature presented him.

[DEVELOP THESE AND OTHER TOPICS; GET FURTHER ROUNDING-OUT QUOTES]

## Goethe and Science

## Goethe and qualities

## All-Sidedness (Goethe)

* Aesthetic of variety
	+ *You must be as mobile and agile as your subject of research.*

## Goethe and Math

## Goethe and Cognition

## Goethe and Music

* Goethe and Mendelssohn

# Chapter 6: The Peirces – Father and Son

# Chapter 6A: Piaget

# Chapter 7: Self-Knowledge

## Foundations of Self-Knowledge – Antonio Damasio

## Octonion Mathematical Interpretation of Damasio Elementary Knowledge Model

## Panksepp and Internal Tracking

## Internet-Based Personal Profiles

# Chapter 8: Artificial Intelligence (AI) and Interior Selves

# Chapter 8A: Levels of Nature and Stages of Mathematical Processing

Pizza Pie Chart



Levels of Nature of Quaternions and Octonions Applications

|  |  |  |
| --- | --- | --- |
| No. | Topic | Primary Researcher or Reference |
| 1 | Quantum mechanics level / Basic Physics | Geoffrey M. Dixon /Dirac |
| 2 | Maxwell Equations / Basic Physics | Doug Sweetser / in progress |
| 3 | Space Shuttle Maneuvering / Astrophysics | *Fill in* |
| 4 | Algebraic Biology – Nucleotides (DNA) – 2 research groups | Sergey PetoukhovAntonio Di Ieva |
| 5 | Molecular Docking | Gwynn Skone |
| 6 | Bio-Logging (animals)Color Processing (primates & humans) | Hassen FouratiRobert Boynton & James Gordon |
| 7 | Animal Emotions and their effects on orientation & navigationAlso Music modeling | Jaak Panksepp, Kevin Behan |
| 8 | Human Cognitive Development | Jean Piaget |

Levels of Nature – quoted characterizations

|  |  |  |
| --- | --- | --- |
| No. | Topic | Quote or Characterization |
| 1 | Quantum/Dixon | *Fill in remainder* |
| 2 | Maxwell Equations/Sweetser |  |
| 3 | Space Shuttle / ? |  |
| 4 | Algebraic Biology (DNA) (1) Petoukhov(2) Antonio Di Ieva | (1) QUOTE FROM ABSTRACT: "Our results testify that living matter possesses a profound algebraic essence. They show new promising ways to develop algebraic biology."--*from Petchoukhov abstract*(2) Even today, the concept of “form” and its quantitation are **not** **...** The form of an object is **defined** in fact when we know its greatness, **....** introduced by the Austrian mathematician Karl **Menger** (1902-1985) .**.....** Fractals **everywhere**. **.....** The genetic code, 8-dimensional **hypercomplex** numbers and dyadic shifts.--*from Google Search* |
| 5 | Molecular Docking / Skone |  |
| 6 | Bio-Logging / FouratiColor / Boynton |  |
| 7 | Animal Emotions – navigation / PankseppMusic in Humans / Behan |  |
| 8 | Human Cognitive Development /Piaget | “Finally, it leads to the attribution of a central role to ordered pairs, in terms of the notions of operation, class, relation , and function.Here we think it is interesting to point out that [C.S.] Peirce was no doubt the first to draw attention to the fundamental role of ordered pairs, which he in fact called ‘elementary relations.’ He showed, in particular, that if we consider four dyadic relations such as ‘colleague of,‘ ‘teacher of,’ ‘student of,’ and ‘classmate of,’ only certain compositions are possible. This led him to the idea, which we consider fundamental to the study of the genesis of intelligence, of an operation which is not everywhere defined. Actually, this is one of the ideas of a hypergroup, as introduced by Menger. Peirce, basing himself on the studies conducted by his father on quaternions, remarked with surprise on the analogy between the table of composition of these four relations [*part of his Theory of Relatives*] and that given by B. Peirce for the quaternions.”*--from J. Piaget (1977, original 1968)* |

## Where We Are Going – Across Levels and Stages

In the next several chapters, we will look at as number of examples of applications of quaternions and octonions at different levels of nature, to extract sub-groups with common properties.

We also will relate these to hypotheses of brain processes that focus on different kinds of math objects and techniques at different stages (typically two or three) of carrying out the overall process.

* Color processing, music processing, and linguistic processing will be explored as domains for deploying such math-operation stages.
* Math application stages include frequency identification, polarities extraction, and global restructuring such as self-dual operations (role swapping).

In this chapter, we look at quaternion and octonion applications and models -- in terms of activities at different levels of nature, including the phenomenon of mathematical and biological imbedding (such as Piagetian imbedding of cognitive schemata, octonion imbedding of quaternions, and “tripartite” human brain architecture in which the reptilian and mammal brains are imbedded in human brains alongside the cortex).

[FILL IN THIS AND NEXT CHAPTERS]

Objects and Contrasts:

[etc]

# Chapter 9: Modern Applications of Quaternions – Selected Examples

## Diversity of Applications of Quaternions and Octonions

1. **Creators/Researchers of Modern Applications of Quaternions**

| **Name** | **Field(s)** | **Dates** | **Comments** |
| --- | --- | --- | --- |
| Sergey Petoukhov | Mechanics of Biology (cells)(Russia) |  | Algebraic biology |
| Hassen Fourati  | Animal Biology, Bio-logging (continuous attitude determination)(France-Grenoble) |  | Articles: H. Fourati, N. Manamanni, A.H.B. Jemaa, L. Afilal, and Y. Handrich, "A quaternion-based Complementary Sliding Mode Observer for attitude estimation: Application in free-ranging animal motions", ;in Proc. CDC, 2010, pp.5056-5061. (IEEE)A rigid-body estimation for bio-logging application: A quaternion-based nonlinear filter approach |
| Lilong Shi, Brian Funt | Color Graphics |  | Paper: Quaternion Color Texture (2005) |
| Gerald Balzano | Music(U.S.) |  | Group theory:Paper: The Group-Theoretic Description of 12-Fold and Microtonal Pitch Systems (Balzano 1980)  |
| Guerino Mazzola | Music | 1947- | Book: Toward a Science of Embodiment (Gestures-based) |
| Geoffrey M. Dixon | Quantum Mechanics |  | Standard ModelBook: Division Algebras: Octonions, quaternions, complex numbers, and the algebraic design of physics, Springer Verlag, 1994 |

1. Applications of Quaternions – Descriptions of Most Salient Uses
	* Development
		+ Embedding
	* Cognition
		+ Logic
		+ Rotation
		+ Interior Selves and Mirror Neurons
	* Communications Control
	* Sub-Atomic Particle Behavior
2. Applications of Quaternions -- Comparative Analysis
3. Octonions and the Relation of Quaternions to Projective Geometry

Interpreting Direction-Finding, Turning, Switching Control, Flow Control

(DEVELOP OUTLINE)

# Chapter 10: Historical Figures (More Figures; Relationships Elaborated)

## Nineteenth Century

* Felix Klein, David Hilbert, and Henri Poincare

## Twentieth Century

* Karl Pribram, Seymour Papert
* David Hestenes, John Baez, and Doug Sweetser
* Jean Piaget and Ben Goertzel

# Chapter 11: Music and Quaternion Modeling

# Chapter 11A: Psychology -- STM (Short-Term Memory), Consciousness, Emotion

# Chapter 12: Color

# Chapter 13: Mathematics

## How the brain deals with math; role of quaternions in this process

## AI Impact on math and understanding and goals of math; role of self-knowledge

## Jack Cowan and the Generation of Mathematical Functions and Patterns

What Numbers, attributes, and relations mean; Range of meaning; Emerson view; Michael Crowe observation: “The Calculus of Quaternions may turn out to be a Calculus of Polarities (2,II: 439-440 – Hamilton volume)”

# Chapter 14: Human Development, Education, and Long-Range Forecast / Self-Knowledge

# Chapter 15: Conclusions

# Bibliography